SOLUTION OF THE PROBLEM OF CURRENT DISTRIBUTION IN A MAGNETOHYDRODYNAMIC CHANNEL WITH PERMEABLE ELECTRODES AT TENSOR CHARACTER OF CONDUCTIVITY OF THE FLOWING MEDIUM

PMM Vol. 32, No. 3, 1968, pp. 353-359

Iu. P. EMETS
(Kiev)

(Received June 5, 1967)

The flow is investigated in a linear magnetohydrodynamic channel with two permeable electrodes for the case of an anisotropic conductivity (stipulated by the Hall effect), when the electroconductive medium is blown in through one electrode and sucked off through the other. This scheme combines the properties of Hall and Faraday energy converters [1].

Mathematical determination of the electric field in the channel leads to the solution of a Riemann – Hilbert inhomogeneous boundary value problem, which by the method of analytic continuation is reduced to a Riemann boundary value problem in the class of automorphic functions.

1. Let us consider a flat magnetohydrodynamic channel $0 \leq y \leq h$, $-\infty < x < \infty$ with two symmetrically placed electrodes which have finite dimensions ab and $a'b'$ $(-l \leqslant x \leqslant l;\ y = 0,\ h)$. We shall assume that the external magnetic field $\mathbf{H}\,(0,\ 0,\ H_z)$ is uniform and perpendicular to the stream $\mathbf{v}\ (u\ (x,\ y),\ v\ (x,\ y),\ 0)$ of the electroconductive medium. For low magnetic Reynolds numbers, as is assumed here, the intrinsic magnetic field of the investigated currents is small compared with the external one, and therefore can be neglected.

In the case of isothermal conditions, according to the phenomenological theory, we have following relations between the density of the electric current, the electric potential and the external force [2].

$$\begin{aligned} j_x(x,\ y) &= \sigma_{xx}(H)\left(-\frac{\partial\varphi}{\partial x} + \frac{1}{c}vH\right) + \sigma_{xy}(H)\left(-\frac{\partial\varphi}{\partial y} - \frac{1}{c}uH\right) \\ j_y(x,\ y) &= \sigma_{yx}(H)\left(-\frac{\partial\varphi}{\partial x} + \frac{1}{c}vH\right) + \sigma_{yy}(H)\left(-\frac{\partial\varphi}{\partial y} - \frac{1}{c}uH\right),\quad H \equiv H_z \end{aligned} \tag{1.1}$$

Here $\sigma_{xx}(H)$, $\sigma_{xy}(H)$, ... are components of the tensor of conductivity, which depend on the magnitude of magnetic field and are interrelated by conditions of symmetry (according to the correlations of Onsager).

$$\sigma_{xx}(H) = \sigma_{yy}(H), \qquad \sigma_{xy}(H) = -\sigma_{yx}(H) \tag{1.2}$$

In general case the components of the conductivity tensor are determined from the kinetic theory.

In particular, for a weakly ionized plasma, we have [3]

$$\sigma_{xx}(H) = \frac{ne^2}{m}\left\langle\frac{\tau}{1 + (e\tau H / mc)^2}\right\rangle, \qquad \sigma_{xy}(H) = \frac{ne^3H}{m^2c}\left\langle\frac{\tau^2}{1 + (e\tau H / mc)^2}\right\rangle \tag{1.3}$$

where n, e, m, and τ are density, charge, mass and relaxation time of electrons. The symbol $\langle\ \rangle$ applied to the function $g(x)$ denotes the integral of the kinetic averaging

$$\langle g(x)\rangle = \frac{4}{3\sqrt{\pi}}\int_0^\infty g(x)\, x^{3/2} e^{-x}\, dx \qquad \left(x = \frac{mv^2}{2kT_e}\right) \tag{1.4}$$

Here ν and T_e are the velocity and temperature of the electrons, k – Boltzmann's constant. If the relaxation time τ, does not depend on ν, according to (1, 4), the angular brackets in (1.3) can be omitted.

For the conditions assumed in Eq. (1.1) together with conditions of continuity of the current and incompressibility of the medium it is possible to introduce the notion of a complex current $j(z)$ expressed by the relations

$$j_x(x, y) = \frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y}, \qquad j_y(x, y) = \frac{\partial P}{\partial y} = -\frac{\partial Q}{\partial x}$$

$$F(z) = P(x, y) + iQ(x, y) \qquad (z = x + iy) \tag{1.5}$$

$$\frac{dF(z)}{dz} = \frac{\partial P}{\partial x} + i\frac{\partial Q}{\partial x} = j_x(x, y) - ij_y(x, y), \qquad \frac{dF(z)}{dz} = j(z)$$

In the band $0 \leqslant \operatorname{Im} z \leqslant h$ the complex current $j(z)$ is determined from the boundary conditions

$$\operatorname{Re}\{(1 + iR_H\sigma(H)H)\,j(t)\} = c^{-1}\sigma(H)Hv(t) \quad \text{on } L' \tag{1.6}$$

$$\operatorname{Im} j(t) = 0 \text{ on } L'', \qquad \lim j(z) = 0 \text{ for } |z| \to \infty \qquad (t \in L = L' + L'')$$

These conditions have been obtained with the assumption that the electrodes and insulators are made from ideal materials. In (1.6) $L' = ab + a'b'$ denotes electrodes and L'' insulators; the conductivity $\sigma(H)$ and Hall constant R_H of the medium flowing in the magnetic field are related with the components of the conductivity tensor by the following expressions:

$$\sigma(H) = \frac{\sigma_{xx}^2(H) + \sigma_{xy}^2(H)}{\sigma_{xx}(H)}, \qquad R_H = \frac{\sigma_{xy}(H)}{H\,[\sigma_{xx}^2(H) + \sigma_{xy}^2(H)]} \tag{1.7}$$

Here the law of the distribution of $v(t)$ on L' is not specified but it is natural to assume that the flow rates of the electroconductive medium at the flow in through one electrode and suck off through the other are finite and equal to each other. In this way we have a magnetohydrodynamic channel with intersecting streams.

The inhomogeneous boundary value problem of Riemann – Hilbert with discontinuity coefficients (the solution of which leads to the determination of the current), will be reduced to the Riemann's boundary value problem in the class of automorphic functions.

For this purpose we shall change the variables by putting

$$\zeta = \pi z / h \qquad (\zeta = \xi + i\eta, \quad z = x + iy) \tag{1.8}$$

This substitution will change the geometrical scale and the points a, b, a' and b' at the edges of the electrodes will acquire new values

$$A = -\pi l / h, \quad B = \pi l / h,$$
$$A' = -\pi l / h + i\pi, \quad B' = \pi l / h + i\pi$$

In the band $\pi < \operatorname{Im} \zeta < 0$ we shall introduce a function $\overline{j(\bar{\zeta})}$, which is the complex-conjugate of $j(\zeta)$ in the region $0 < \operatorname{Im} \zeta < \pi$ at the points symmetrical with respect to the axis ξ.

Now the boundary conditions (1.6) are reduced to the following form:

$$\Psi^+(t_1) = -\frac{1 - iR_H\sigma(H)H}{1 + iR_H\sigma(H)H}\Psi^-(t_1) + \frac{2\sigma(H)v(t_1)H}{c(1 + iR_H\sigma(H)H)} \quad \text{on } L_1'$$

$$\Psi^+(t_1) = \Psi^-(t_1) \quad \text{on } L_1'', \qquad \lim \Psi(\zeta) = 0 \quad \text{for } |\zeta| \to \infty$$

$$(t_1 = L_1 = L_1' + L_1'')$$

$$\Psi(\zeta) = \begin{cases} \Psi^+(\zeta) & \text{for } 0 < \operatorname{Im}\zeta < \pi, \qquad \Psi^+(\zeta) = j(\zeta) = j_\xi(\xi, \eta) - ij_\eta(\xi, \eta) \\ \Psi^-(\zeta) & \text{for } -\pi < \operatorname{Im}\zeta < 0, \qquad \Psi^-(\zeta) = \overline{j(\bar{\zeta})} = j_\xi(\xi, -\eta) + ij_\eta(\xi, -\eta) \end{cases} \quad (1.9)$$

Here $L_1' = AB + A'B' + A''B''$ $(A'' = -\pi l / h - i\pi,\ B'' = \pi l/j - i\pi)$, and L_1'' is the remaining part of the straight lines $\eta = 0$ and $\eta = |\pi|$.

The band $-\pi < \operatorname{Im} \zeta < \pi$ is one of the fundamental regions of a single period group, created by the parabolic substitutions $\mu_k(\zeta) = \zeta + 2\pi ik$ $(k \pm 1, \pm 2, \ldots)$. The function e^ζ is automorphic in relation to the group μ_k, this function has a first order pole on the right end of the band.

Eqs. (1.9) represent the Riemann boundary value problem with discontinuity coefficients in the class of periodic functions, where $v(t)$ must satisfy Hölder boundary condition on L_1'.

The most general solution of the problem (1.9) which satisfies the initial conditions (1.6) has the form (everywhere in the following the previous variables are used):

$$\Psi(z) = [(e^{\pi z/h} - e^{-\pi l/h})(e^{\pi z/h} + e^{\pi l/h})]^{-1/2+\varepsilon} [(e^{\pi z/h} + e^{-\pi l/h})(e^{\pi z/h} - e^{\pi l/h})]^{-1/2-\varepsilon} \times$$

$$\times \left\{ \frac{\sigma(H)H}{\pi c\sqrt{1 + (R_H\sigma(H)H)^2}} \int_{L'} [(e^t - e^{-\pi l/h})(e^t + e^{\pi l/h})]^{1/2-\varepsilon} [(e^t + e^{-\pi l/h}) \times \right.$$

$$\left. \times (e^{\pi l/h} - e^t)]^{1/2+\varepsilon} \frac{v(t)e^t\,dt}{e^t - e^{\pi z/h}} + Ce^{\pi z/h} \right\} \quad (t \in L', -h < \operatorname{Im} z < h) \quad (1.10)$$

$$\varepsilon = \pi^{-1} \operatorname{arctg} R_H\sigma(H)H, \quad 0 \leqslant \varepsilon \leqslant 1/2$$

$$\Psi(z) = Ce^{-\pi z/h} + O(e^{-2\pi z/h}) \quad \text{for } |z| \to \infty$$

Where

$$\Psi(z) = \begin{cases} j(z) = j_x(x, y) - ij_y(x, y) & \text{for } 0 < \operatorname{Im} z < h \\ \overline{j(\bar{z})} = j_x(x, -y) + ij_y(x, -y) & \text{for } -h < \operatorname{Im} z < 0 \end{cases}$$

Function $\Psi(z)$ is limited on the edges of the band $-h < \operatorname{Im} z < h$ and possesses integrable properties at the ends of electrodes a, b, a' and b'. The real constant C in Eq. (1.10), can be determined from additional physical considerations.

2. Now we shall calculate the basic integral characteristics of the channel. For the sake of definiteness we shall assume that the magnetohydrodynamic channel is working in the generator mode.

Using Sokhotskii – Plemel formulas

$$\Psi^{+}(t)=\frac{\sigma(H)Hv(t)}{c(1+iR_H\sigma(H)H)}+\Pi_1(t)\left\{\frac{\sigma(H)H}{\pi ic(1+iR_H\sigma(H)H)}\times\right. \tag{2.1}$$

$$\left.\times\int_{L'}\frac{v(s)}{\Pi_1(s)}\frac{e^s ds}{e^s-e^t}-\frac{i(1-iR_H\sigma(H)H)}{\sqrt{1+(R_H\sigma(H)H)^2}}Ce^t\right\}(s,t\in L')$$

$$\Psi^{-}(t)=\frac{\sigma(H)Hv(t)}{c(1-iR_H\sigma(H)H)}-\Pi_1(t)\left\{\frac{\sigma(H)H}{\pi ic(1-iR_H\sigma(H)H)}\times\right.$$

$$\left.\times\int_{L'}\frac{v(s)}{\Pi_1(s)}\frac{e^s ds}{e^s-e^t}-\frac{i(1+iR_H\sigma(H)H)}{\sqrt{1+(R_H\sigma(H)H)^2}}Ce^t\right\}(s,t\in L')$$

$$\Pi_1(t)=[(e^t-e^{-\pi l/h})(e^t+e^{\pi l/h})]^{-1/2+\varepsilon}[(e^t+e^{-\pi l/h})(e^{\pi l/h}-e^t)]^{-1/2-\varepsilon}$$

we find the values of the normal and tangential components of the current on the electrodes:

$$j_x(t)=\frac{\Psi^{+}(t)+\Psi^{-}(t)}{2}=\frac{\sigma(H)Hv(t)}{c[1+(R_H\sigma(H)H)^2]}-\frac{R_H\sigma(H)H\Pi_1(t)}{\sqrt{1+(R_H\sigma(H)H)^2}}\times$$

$$\times\left\{\frac{\sigma(H)H}{\pi c\sqrt{1+(R_H\sigma(H)H)^2}}\int_{L'}\frac{v(s)}{\Pi_1(s)}\frac{e^s ds}{e^s-e^t}+Ce^t\right\}(s,t\in L') \tag{2.2}$$

$$j_y(t)=\frac{\Psi^{-}(t)-\Psi^{+}(t)}{2i}=\frac{R_Hv(t)(\sigma(H)H)^2}{c[1+(R_H\sigma(H)H)^2]}+\frac{\Pi_1(t)}{\sqrt{1+(R_H\sigma(H)H)^2}}\times$$

$$\times\left\{\frac{\sigma(H)H}{\pi c\sqrt{1+(R_H\sigma(H)H)^2}}\int_{L'}\frac{v(s)}{\Pi_1(s)}\frac{e^s ds}{e^s-e^t}+Ce^t\right\}(s,t\in L')$$

Constant C is determined in this case from the Ohm's integral law applied to the channel and the outer circuit.

$$E-2\varphi_e=I\Omega_H \tag{2.3}$$

Here E is the electromotive force; $2\varphi_e$ is the potential between the electrodes; I is the total current, passing through the electrodes and Ω_H is the external load.

Taking into account that on insulators $j_y=0$ we obtain on the axis x

$$\frac{\partial\varphi}{\partial x}-\frac{1}{c}vH=\frac{j_x(x)}{\sigma(H)}=\frac{H\Pi_2(x)}{\pi c\sqrt{1+(R_H\sigma(H)H)^2}}\int_{L^1}\frac{v(t)}{\Pi_2(t)}\frac{e^t dt}{e^t-e^x}+\frac{C}{\sigma(H)}e^x\Pi_2(x) \tag{2.4}$$

$$E-2\varphi_e=\left(\int_{-\infty}^{-\lambda}+\int_{\lambda}^{\infty}\right)\left(\frac{\partial\varphi}{\partial x}-\frac{1}{c}vH\right)_{y=0}dx,\quad E=\frac{1}{c}HG,\quad I=\delta\int_{-\lambda}^{\lambda}j_y(x)\,dx$$

$$\Pi_2(x)=[(e^x-e^{-\lambda})(e^x+e^{\lambda})]^{-1/2+\varepsilon}[(e^x+e^{-\lambda})(e^x-e^{\lambda})]^{-1/2-\varepsilon}\quad\left(\lambda=\frac{\pi l}{h}\right)$$

Here δ is the height of the channel, G is the volume flow-rate of the medium of the main stream.

After substituting (2.4) into (2.3) and a simple transformation we obtain

$$C=\left\{E-\frac{H(\Delta_4+\Delta_5)}{\pi c[1+(R_H\sigma(H)H)^2]}-\frac{1}{c}\delta\sigma(H)H\left(R_H\sigma(H)H\Delta_1+\frac{\Delta_2}{\pi}\right)\Omega_H\right\}:$$

$$:\left\{\frac{\Delta_6+\Delta_7}{\sigma(H)}+\frac{\delta\Delta_3\Omega_H}{\sqrt{1+(R_H\sigma(H)H)^2}}\right\} \tag{2.5}$$

$$\Delta_0(t) = \int_{L'} \frac{v(s)}{\Pi_1(s)} \frac{e^s ds}{e^s - e^t}, \qquad \Delta_1 = \int_{-\lambda}^{\lambda} v(t)\,dt, \qquad \Delta_2 = \int_{-\lambda}^{\lambda} \Pi_1(t)\,\Delta_0(t)\,dt$$

$$\Delta_3 = \int_{-\lambda}^{\lambda} e^t \Pi_1(t)\,dt, \qquad \Delta_4 = \int_{-\infty}^{-\lambda} \Pi_2(t)\,\Delta_0(t)\,dt, \qquad \Delta_5 = \int_{\lambda}^{\infty} \Pi_2(t)\,\Delta_0(t)\,dt$$

$$\Delta_6 = \int_{-\infty}^{-\lambda} e^t \Pi_2(t)\,dt, \qquad \Delta_7 = \int_{\lambda}^{\infty} e^t \Pi_2(t)\,dt \qquad \left(\lambda = \frac{\pi l}{h}\right) \tag{2.6}$$

In order to clarify further theoretical computation and the explanation of physical effects, stipulated by the simultaneous effect of the anisotropy of conductivity and the presence of permeable electrodes, we shall first consider two particular modes of the converter operation : idle running and short circuit.

In the first case $I = 0$ and we have

$$2\varphi_e = 2\varphi_{xx} = E - \frac{H(\Delta_4 + \Delta_5)}{\pi c\,[1 + (R_H \sigma(H) H)^2]} + \sqrt{1 + (R_H \sigma(H) H)^2} \times$$

$$\times \frac{\Delta_6 + \Delta_7}{\sigma(H)\,\Delta_3} \left(\frac{1}{c} R_H \Delta_1 (\sigma(H) H)^2 + \frac{\sigma(H) H}{\pi c} \Delta_2\right) = (1 + \Lambda) E = E_* \tag{2.7}$$

It follows from this that the electromotive forces in the channels with permeable and impermeable electrodes differ by the factor $(1 + \Lambda)$. In the scheme of flow considered, the dimensionless parameter assumes zero value at $v(t) = 0$ on L' (no flow in or out through the electrodes) and when $R_H = 0$ (Hall effect is absent) the sign of Λ depends on the mutual direction of the main stream and blown stream in the channel.

At the short circuit conditions $2\varphi_e = 0$ and therefore

$$I = I_* = \frac{\delta \sigma(H)\,\Delta_3}{(\Delta_6 + \Delta_7)\sqrt{1 + (R_H \sigma(H) H)^2}} \left\{E - \frac{H(\Delta_4 + \Delta_5)}{\pi c\,[1 + (R_H \sigma(H) H)^2]} + \right.$$

$$\left. + \sqrt{1 + (R_H \sigma(H) H)^2}\, \frac{\Delta_6 + \Delta_7}{\sigma(H)\,\Delta_3} \left(\frac{1}{c} R_H \Delta_1 (\sigma(H) H)^2 + \frac{\sigma(H) H}{\pi c} \Delta_2\right)\right\} =$$

$$= \frac{(1 + \Lambda) E}{\Omega_b} = \frac{E_*}{\Omega_b} \tag{2.8}$$

$$\Omega_b = \sqrt{1 + (R_H \sigma(H) H)^2}\, \frac{\Delta_6 + \Delta_7}{\delta \sigma(H) \Delta_3}$$

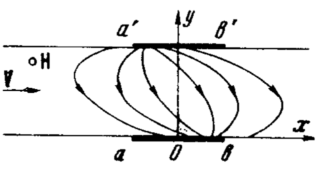

Fig. 1

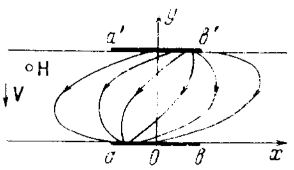

Fig. 2

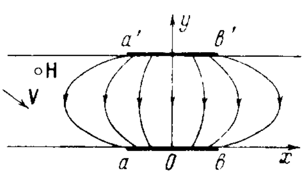

Fig. 3

The magnitude Ω_b represents the internal resistance of the generator with impermeable, solid electrodes. The value of Ω_b was theoretically investigated in sufficient detail for a wide range of change of physical constants and the channel geometry [6 to 8].

In the general case, if the load parameter q (characterising the regime of the work of the power converter)

$$q = \frac{\Omega_{\text{н}}}{\Omega_{\text{н}} + \Omega_b} \qquad 0 \leqslant q \leqslant 1 \tag{2.9}$$

is introduced, the formulas for the total current I and electrical power N assume a form similar to the equations for the channel with impermeable electrodes:

$$I = (1 - q)(1 + \Lambda)\frac{E}{\Omega_b} = (1 - q)\frac{E_*}{\Omega_b}$$

$$N = 2\varphi_e I = q(1 - q)(1 + \Lambda)^2 \frac{E^2}{\Omega_b} = q(1 - q)\frac{E_*^2}{\Omega_b} \tag{2.10}$$

Keeping in mind that the components of velocity for an incompressible medium are expressed by the current function $\psi(x, y)$ $(u = \partial\psi / \partial y,\ v = -\partial\psi / \partial x)$, we find the Joule dissipation

$$Q = \int_{-\infty}^{\infty}\int_0^h \frac{\mathbf{j}^2}{\sigma(H)}\,dxdy = \int_{-\infty}^{\infty}\int_0^h \left\{-\mathbf{j}\operatorname{grad}\varphi + \frac{1}{c}\mathbf{j}(\mathbf{v}\times\mathbf{H})\right\}dxdy =$$

$$= \int_{-\infty}^{\infty}\int_0^h \operatorname{div}\left[(\varphi(x, y) + \frac{1}{c}H\psi(x, y))\,\mathbf{j}\right]dxdy = \tag{2.11}$$

$$= \frac{1}{c}H\int_{-\lambda}^{\lambda}\psi(x, 0)\,j_y(x, 0)\,dx - \frac{1}{c}H\int_{\lambda}^{-\lambda}\psi(x, h)\,j_y(x, h)\,dx - N \qquad \left(\lambda = \frac{\pi l}{h}\right)$$

If there is no injection of the medium, Eqs. (2.10) and (2.11) change into known relations [6 to 8]

$$I = (1 - q)\,E / \Omega_b, \qquad N = q(1 - q)\,E^2 / \Omega_b, \qquad Q = EI - N$$

A qualitative picture of the electric field in the channel with permeable electrodes can be presented in an approximate form as a superposition of two partial distributions of the current.

When there is no in- and outflow the lines of current are distorted in the middle zone of the channel and basically concentrated in small regions at the ends of electrodes (Fig. 1). This phenomenon is connected with screening of the Hall's emf acting along the channel by the conductive walls of the channel.

If the basic stream is not present and only the conductive medium is injected through one electrode and sucked off through the other (for example, with constant speed) then a Faraday emf is generated along the channel. This emf produces circulating currents on the electrodes, this, in turn, leads to the appearance of Hall's emf which, unlike in the previous case, acts now between the electrodes. Current lines for this kind of flow are presented in Fig. 2.

The resultant qualitative picture of the current distribution in the channel with permeable electrodes is presented in Fig. 3.

In the scheme of the channel under consideration it is possible to control the current distribution on the electrodes by choosing a suitable rule for the injection and suction out of the stream, and by choosing the ratio of the flow rate of the basic and the injected streams, it is possible to adjust the total current which passes through the load, and the potential between the electrodes.

BIBLIOGRAPHY

1. Emets, Iu. P., On the distribution of current on the permeable electrodes in the presence of Hall effect in a stream of electroconductive medium. PMTF, No. 3 (pp 35-41), 1966.
2. Landau, L.D. and Lifchits, E.M., Electrodynamics of Continuous Media. M., Gostekhizdat, 1957.
3. Ginzburg, V.L. and Gurevich, A.B., Nonlinear phenomena in plasma in an alternating electromagnetic field. Usp. fiz. nauk., Vol. 70, No. 2, 1960.
4. Gakhov, F.D., Boundary Value Problems. M., Fizmatgiz., 1963.
5. Chibrikova, L. I., On the Riemann's boundary value problem for the automorphic functions. Uch. zapiski Kazan. Univ Vol. 116, Book No. 4, 1956.
6. Emets, Iu. P., Distribution of the current in a flat magnetohydrodynamic channel at the flow of the electroconductive medium in a strong magnetic field. PMTF, No. 3, (pp 3-11), 1967.
7. Vatazhin, A.B., Some two-dimensional problems on the current distribution in an electroconductive medium flowing in a channel in a magnetic field. PNTF, No. 2, (pp 39-54), 1963.
8. Shultz – Grunov, F. and Danzel, D.L., Calculation of the electric characteristics of an MHD generator with finite electrodes by conformal mapping under full consideration of tensor conductivity and a velocity profile. Intern. Sympos. Magnetohydrodynamic generation, Paris, Vol. 2, 1964.

Translated by H. B.